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Stability Theorems of Perturbed Linear Ordinary Differential Equations

TAKESHI TANIGUCHI

Department of Mathematics, Kurume University, Miimachi, Kurume, Fukuoka, Japan

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In this paper, we present a sufficient condition to ensure that if the zero solution of a linear differential equation is uniform-asymptotically stable, then the zero solution of its perturbed linear differential equation is also uniform-asymptotically stable. Our theorem is a generalization of Perron's celebrated theorem and is proved by the comparison theorem of integral inequalities. © 1990 Academic Press, Inc.

1. INTRODUCTION

The qualitative investigation of solutions of linear differential equations and their perturbed linear differential equations plays a very important role for handling various problems on mechanical, electronic, control engineering, or economic, and other practical problems. Thus a number of authors have been studying many problems concerned with them and presenting numerous properties.

Among such properties, the following Perron's theorem is one of the most famous theorems.

THEOREM A. *Let the perturbed linear differential equation* (A): $dx/dt = A(t)x + f(t, x)$ *satisfy the following conditions*:

(a) $A(t)$ *is continuous and bounded on* $0 \leqslant t < \infty$,

(b) $f(t, x)$ *is continuous on* $[0, \infty) \times B_K$, $f(t, 0) \equiv 0$,

(c) $\|f(t, x)\| = o(\|x\|)$ *uniformly in* t.

If the zero solution of the linear differential equation $dx/dt = A(t)x$ *is uniform-asymptotically stable, then the zero solution of* (A) *is also uniform-asymptotically stable.*

In a sense, it is natural that, for stable linear differential equations, the perturbed linear differential equations by very small perturbations are also

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stable. However, when linear differential equations are considered, if large perturbations would be given, are their perturbed differential equations stable? It is a very important and interesting problem under what conditions the stability is ensured.

The purpose of this paper is to give an answer for this question and to generalize the above Theorem A by the variation-of-constants formula and the comparison theorem of integral inequalities.

2. Preliminaries

Let R^n and R^+ be the n-dimensional Euclidian space and the set of all non-negative real numbers, respectively. $C[X, Y]$ denotes the set of all continuous mappings from a topological space X to a topological space Y. Set $B_\delta = \{x \in R^n \mid \|x\| < \delta\}$ for a positive real number δ where $\|\cdot\|$ denotes a usual norm. Let $A(t)$ be a continuous $n \times n$ matrix defined on $[0, \infty)$ and let $f(t, x) \in C[[0, \infty) \times R^n, R^n]$.

Consider a linear differential equation

$$dx/dt = A(t)x \tag{1}$$

and a perturbed differential equation of (1)

$$dx/dt = A(t)x + f(t, x). \tag{2}$$

Let $U(t)$ be the fundamental matrix solution of (1). Then the solution $x(t)$ of (2) satisfies the integral equation

$$x(t) = U(t)\, U^{-1}(t_0)\, x(t_0) + \int_{t_0}^{t} U(t)\, U^{-1}(s)\, f(s, x(s))\, ds, \qquad t \geqslant t_0.$$

Now we give stability definition of the zero solution of a differential equation

$$dy/dt = g(t, y), \tag{3}$$

where $g \in C[[0, \infty) \times R^n, R^n]$. Let $y(t) \equiv y(t; t_0, y_0)$ denote a solution of (3) with an initial value (t_0, y_0).

Definition 1. The zero solution of (3) is said to be

(S) stable if for any $\varepsilon > 0$ and any $t_0 \geqslant 0$, there exists a $\delta(t_0, \varepsilon) > 0$ such that if $\|y(t_0)\| < \delta(t_0, \varepsilon)$, then $\|y(t; t_0, y_0)\| < \varepsilon$ for all $t \geqslant t_0$,

(US) uniformly stable if the $\delta(t_0, \varepsilon)$ in (S) is independent of time t_0, (QEAS) quasi-equiasymptotically stable if for any $\varepsilon > 0$ and any $t_0 \geq 0$, there exist a $\delta(t_0) > 0$ and a $T(t_0, \varepsilon) > 0$ such that if $\|y(t_0)\| < \delta(t_0)$, then $\|y(t; t_0, y_0)\| < \varepsilon$ for all $t \geq t_0 + T(t_0, \varepsilon)$,

(QUAS) quasi-uniform-asymptotically stable if the $\delta(t_0)$ and the $T(t_0, \varepsilon)$ in (QEAS) are independent of time t_0,

(UAS) uniformly asymptotically stable if it is uniformly stable and is quasi-uniform-asymptotically stable.

Next we give boundedness definition of solutions of (3).

DEFINITION 2. The solutions of (3) are said to be

(B) equibounded if for any $\rho > 0$ and any $t_0 \geq 0$, there exists a $\beta(t_0, \rho) > 0$ such that if $\|y(t_0)\| < \rho$, then $\|y(t; t_0, y_0)\| < \beta(t_0, \rho)$ for all $t \geq t_0$,

(UB) uniformly bounded if the $\beta(t_0, \rho)$ in (B) is independent of time t_0,

(EUB) equiultimately bounded if there exists a $B > 0$, and for any $\rho > 0$ and any $t_0 \geq 0$, there exists a $T(t_0, \rho) > 0$ such that if $\|y(t_0)\| < \rho$, then $\|y(t; t_0, y_0)\| < B$ for all $t \geq t_0 + T(t_0, \rho)$,

(UUB) uniform-ultimately bounded if the $T(t_0, \rho)$ in (EUB) is independent of time t_0.

At the end of this section, we present a lemma for integral inequalities This lemma is a key lemma for our theorems.

LEMMA 1 [15, p. 315]. *Let the following condition* (A) *or* (B) *hold for functions* $f(t)$, $g(t) \in C[[t_0, \infty), R^+]$ *and* $F(t, u) \in C[[t_0, \infty) \times R^+, R^+]$:

(A) $f(t) - \int_{t_0}^t F(s, f(s))\, ds \leq g(t) - \int_{t_0}^t F(s, g(s))\, ds$, $t \geq t_0$ *and* $F(s, u)$ *is strictly increasing in u for each fixed* $s \geq 0$.

(B) $f(t) - \int_{t_0}^t F(s, f(s))\, ds < g(t) - \int_{t_0}^t F(s, g(s))\, ds$, $t \geq t_0$ *and* $F(s, u)$ *is monotone nondecreasing in u for each fixed* $s \geq 0$.

If $f(t_0) < g(t_0)$, then $f(t) < g(t)$, $t \geq t_0$.

3. STABILITY THEOREMS

In this section we discuss the stability of the zero solution of the differential equation (2) and throughout this section assume that $f(t, 0) \equiv 0$. Let B_δ^+ denote the set $\{u \in R^+ \mid 0 \leq u < \delta\}$. To begin with, the following theorem is proved.

THEOREM 1. *Let the following conditions hold for the differential equation* (2):

(1a) $\|f(t,x)\| \leqslant F(t, \|x\|)$, $F(t,0) \equiv 0$, *and* $F(t,u)$ *is monotone nondecreasing with respect to* u *for each fixed* $t \geqslant 0$,

(1b) $F(t,u) \in C[[0,\infty) \times B_\delta^+, R^+]$,

(1c) *the zero solution of the differential equation* (1) *is uniformly stable, that is, there exists a constant* $K \geqslant 1$ *such that* $\|U(t)\,U^{-1}(s)\| \leqslant K$, $t \geqslant s \geqslant 0$.

If the zero solution of the differential equation

$$dy/dt = KF(t, y) \tag{4}$$

is uniformly stable (*stable, uniform-asymptotically stable*), *then the zero solution of* (2) *is also uniformly stable* (*stable, uniform-asymptotically stable*).

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ be a solution of (2) with an initial value (t_1, x_1), $t_1 \geqslant 0$. Then the solution $x(t)$ is of the form

$$x(t) = U(t)\,U^{-1}(t_1)\,x_1 + \int_{t_1}^{t} U(t)\,U^{-1}(s)\,f(s, x(s))\,ds. \tag{5}$$

Thus we obtain from conditions (1c) and (1a) that

$$\begin{aligned}\|x(t)\| &\leqslant K\,\|x_1\| + \int_{t_1}^{t} K\,\|f(s, x(s))\|\,ds \\ &\leqslant K\,\|x_1\| + \int_{t_1}^{t} KF(s, \|x(s)\|)\,ds.\end{aligned}$$

Next let $y(t) \equiv (t; t_1, y_1)$ be the solution of (4) passing through (t_1, y_1) and let $K\,\|x_1\| < y_1$. Then

$$\|x(t)\| - \int_{t_1}^{t} KF(s, \|x(s)\|)\,ds < y(t) - \int_{t_1}^{t} KF(s, y(s))\,ds.$$

Therefore applying Lemma 1, we obtain that

$$\|x(t)\| < y(t), \qquad t \geqslant t_1.$$

Since the zero solution of (4) is uniformly stable, for any $\varepsilon > 0$ there exists a $\delta_1(\varepsilon) > 0$ such that if $|y_1| < \delta_1(\varepsilon)$, then $|y(t)| < \varepsilon$ for all $t \geqslant t_1$. Thus set $\delta(\varepsilon) \equiv \delta_1(\varepsilon)/K$. If $\|x(t_1)\| < \delta(\varepsilon)$, then take a $y_1 > 0$ such that

$K\|x_1\| < y_1 < \delta_1(\varepsilon)$. Therefore we have that $\|x(t)\| < \varepsilon$ for all $t \geq t_1$, which completes the proof of the theorem.

As the corollary, the following well-known property is proved.

COROLLARY 1. *Let the following conditions hold for the differential equation* (2):

(1d) $\|f(t, x)\| \leq a(t)\|x\|$,

(1e) $a(s) \in C[[0, \infty), R^+]$ *and there exists a positive constant* M *such that* $\int_0^\infty a(s)\, ds < M$.

If condition (1c) *holds, then the zero solution of* (2) *is uniformly stable.*

Proof. Set $F(t, u) \equiv a(t)u$, $u \geq 0$. First of all, we show that the zero solution of the differential equation

$$dy/dt = Ka(t)y \tag{6}$$

is uniformly stable. Let $y(t) \equiv y(t; t_1, y_1)$, $t \geq t_1 \geq 0$ be a solution of (6) passing through (t_1, y_1). Then we obtain that

$$y(t) \leq y(t_1)\, e^{KM},$$

which implies that the zero solution of (6) is uniformly stable. Therefore since conditions (1a), (1b), and (1c) of Theorem 1 are satisfied, the zero solution of (2) is uniformly stable.

Here we give the following new stability definition:

DEFINITION 3. Let an $m > 0$ denote a positive real number. The zero solution of the differential equation (3) is said to be $T(m)$-asymptotically stable if for any $t_1 \geq 0$ and any $\varepsilon > 0$, there exist a $T(t_1, \varepsilon) > 0$ and a $\delta_0(t_1) > 0$ such that

$$\|y(t)\, e^{-m(t-t_1)}\| < \varepsilon,\ t \geq t_1 + T(t_1, \varepsilon) \text{ provided } \|y(t_1)\| < \delta_0(t_1).$$

The zero solution of (3) is said to be $T(m)$-uniform-asymptotically stable if the above $T(t_1, \varepsilon)$ and $\delta_0(t_1)$ are independent of time t_1.

LEMMA 2. *If the zero solution of* (3) *is uniformly stable, then it is* $T(m)$-*uniform-asymptotically stable.*

Proof. Let a $B > 0$ be any fixed positive real number. By the assumption of the lemma there exists a $\delta(B) > 0$ such that if $\|y(t_1)\| < \delta(B)$, then $\|y(t)\| < B$ for all $t \geq t_1$. Thus let any $\varepsilon > 0$ be given. We may assume that $\varepsilon < B$. Set $T(\varepsilon) \equiv (1/m) \log(B/\varepsilon)$. Hence, if $\|y(t_1)\| < \delta(B)$, then $\|y(t)\, e^{-m(t-t_1)}\| < \varepsilon$, $t \geqslant t_1 + T(\varepsilon)$, which completes the proof of the lemma.

LEMMA 3. *If the solutions of* (3) *are uniformly bounded, then the zero solution of* (3) *is $T(m)$-uniform-asymptotically stable.*

Proof. Let a $B > 0$ be any fixed positive real number. By the assumption of the lemma, there exists a $\beta(B) > 0$ such that if $\|y(t_1)\| < B$, then $\|y(t)\| < \beta(B)$ for all $t \geqslant t_1$. Thus let any $\varepsilon > 0$ be given. We assume that $\varepsilon < B$. Set $T(\varepsilon) \equiv (1/m) \log(B/\varepsilon)$. Hence, if $\|y(t_1)\| < \delta(B)$, then $\|y(t)\, e^{-m(t-t_1)}\| < \varepsilon$, $t \geqslant t_1 + T(\varepsilon)$, which completes the proof of the lemma.

LEMMA 4. *If the zero solution of* (3) *is stable or if the solutions of* (3) *are equibounded, then the zero solution of* (3) *is $T(m)$-asymptotically stable.*

LEMMA 5. *If the zero solution of* (3) *is quasi-uniform-asymptotically stable, then it is $T(m)$-uniform-asymptotically stable.*

Proof. Let a $B > 0$ be any fixed positive real number. By the assumption of the lemma, there exist a δ_0 and a $T_0(B) > 0$ such that if $\|y(t_1)\| < \delta_0$, then $\|y(t)\| < B$ for all $t \geqslant t_1 + T_0(B)$. Thus let any $\varepsilon > 0$ be given. We assume $\varepsilon < B$. Set $T(\varepsilon) \equiv T_0(B) + (1/m) \log(B/\varepsilon)$. Hence if $\|y(t_1)\| < \delta_0$, then $\|y(t)\, e^{-m(t-t_1)}\| < \varepsilon$, $t \geqslant t_1 + T(\varepsilon)$, which completes the proof of the lemma.

LEMMA 6. *If the solutions of* (3) *are uniform-ultimately bounded, then the zero solution of* (3) *is $T(m)$-uniform-asymptotically stable.*

LEMMA 7. *If the zero solution of* (3) *is quasi-equiasymptotically stable or if the solutions of* (3) *are equiultimately bounded, then the zero solution of* (3) *is $T(m)$-asymptotically stable.*

Next we discuss the quasi-equiasymptotic stability of the zero solution of the differential equation (2).

THEOREM 2. *Let conditions* (1a) *and* (1b) *hold for the differential equation* (2). *Furthermore suppose that the following conditions are satisfied*:

(2a) *the zero solution of the differential equation* (1) *is uniformly asymptotically stable, that is, there exist constants $K \geqslant 1$ and $m > 0$ such that* $\|U(t)U^{-1}(s)\| \leqslant Ke^{-m(t-s)}$, $0 \leqslant s \leqslant t$,

(2b) $\alpha F(s, u) \leqslant F(s, \alpha u)$ *for any real number $\alpha \geqslant 1$ and any $u \in R^+$*,

If the zero solution of (4) *is $T(m)$-asymptotically stable, then the zero solution of* (2) *is quasi-equiasymptotically stable.*

Proof. Let any $\varepsilon > 0$ and any $t_1 \geqslant 0$ be given and let $x(t) \equiv x(t; t_1, x_1)$ and $y(t) \equiv y(t; t_1, y_1)$ be solutions of (2) and (4), respectively. By the assumption of the theorem, there exist a $T(t_1, \varepsilon) > 0$ and a $\delta_0(t_1) > 0$ such that $|y(t)\, e^{-m(t-t_1)}| < \varepsilon$ for all $t \geqslant t_1 + T(t_1, \varepsilon)$ provided $|y_1| < \delta_0(t_1)$. From (5), conditions (1a) and (2a), we obtain that

$$\|x(t)\| \leqslant Ke^{-m(t-t_1)}\, \|x_1\| + K \int_{t_1}^{t} e^{-m(t-s)} F(s, \|x(s)\|)\, ds.$$

Thus by condition (2b),

$$\|x(t)\|\, e^{mt} \leqslant Ke^{m(t_1)}\, \|x_1\| + \int_{t_1}^{t} KF(s, \|x(s)\|\, e^{ms})\, ds. \tag{7}$$

Let $Ke^{m(t_1)}\, \|x_1\| < y_1$. Then, we have that

$$\|x(t)\|\, e^{mt} - \int_{t_1}^{t} KF(s, \|x(s)\|\, e^{ms})\, ds < y(t)$$

$$- \int_{t_1}^{t} KF(s, y(s))\, ds.$$

Hence by Lemma 1, $\|x(t)\|\, e^{mt} < y(t)$, that is, $\|x(t)\| < y(t)\, e^{-mt}$ for all $t \geqslant t_1$. Now set $\delta(t_1) \equiv \delta_0(t_1)/(Ke^{mt_1})$. Thus if $\|x(t_1)\| < \delta(t_1)$, then take a $y_1 > 0$ such that $Ke^{m(t_1)}\, \|x_1\| < y_1 < \delta_0(t_1)$. Therefore, we obtain that

$$\|x(t)\| < y(t)\, e^{-mt} < y(t)\, e^{-m(t-t_1)} < \varepsilon, \qquad t \geqslant t_1 + T(t_1, \varepsilon),$$

which completes the proof of the theorem.

COROLLARY 2.1. *Let conditions* (1a), (1b), (2a), *and* (2b) *hold for the differential equation* (2). *If the zero solution of* (4) *is stable or quasi-equiasymptotically stable, or if the solutions of* (4) *are equibounded or equiultimately bounded, then the zero solution of* (2) *is quasi-equiasymptotically stable.*

Proof. By Lemma 4 or 7, the zero solution of (2) is $T(m)$-asymptotically stable. Thus by Theorem 2, the proof is complete.

THEOREM 3. *Let conditions* (1a), (1b), *and* (2a) *be satisfied for the differential equation* (2) *and let the following condition hold*:

(3a) $\alpha F(t, u) = F(t, \alpha u)$ *for any real number* $\alpha \geqslant 1$ *and any* $u \in R^+$.

If the zero solution of (4) *is* $T(m)$*-uniform-asymptotically stable, then the zero solution of* (2) *is quasi-uniform-asymptotically stable.*

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ and $y(t) \equiv y(t; t_1, y_1)$ be solutions of (2) and (4), respectively. From condition (3a),

$$y(t)\, e^{m(t_1)} = y_1\, e^{m(t_1)} + \int_{t_1}^{t} KF(s, e^{m(t_1)}\, y(s))\, ds.$$

Let $K\, \|x_1\| < y_1$. Then from (7), we obtain that

$$\|x(t)\|\, e^{mt} - \int_{t_1}^{t} KF(s, \|x(s)\|\, e^{ms})\, ds < y(t)\, e^{m(t_1)}$$
$$- \int_{t_1}^{t} KF(s, y(s)\, e^{m(t_1)})\, ds.$$

Therefore by Lemma 1, $\|x(t)\|\, e^{mt} < y(t)\, e^{mt_1}$, that is, $\|x(t)\| < y(t)\, e^{-m(t-t_1)}$ for all $t \geqslant t_1$. Let any $\varepsilon > 0$ be given. Then by the assumption, there exist a $\delta_0 > 0$ and a $T(\varepsilon) > 0$ such that if $|y_1| < \delta_0$, then $|y(t)\, e^{-m(t-t_1)}| < \varepsilon$ for all $t \geqslant t_1 + T(\varepsilon)$. Thus set $\delta \equiv \delta_0/K$. If $\|x_1\| < \delta$, then take a $y_1 > 0$ such that $K\, \|x_1\| < y_1 < \delta_0$. Therefore,

$$\|x(t)\| < y(t)\, e^{-m(t-t_1)} < \varepsilon, \qquad t \geqslant t_1 + T(\varepsilon),$$

which completes the proof of the theorem.

As the corollaries of Theorem 3, we prove the following well known properties.

COROLLARY 3.1. *Let the following conditions hold for the differential equation* (2):

(1d) $\|f(t, x)\| \leqslant a(t)\, \|x\|$,

(1e) $a(t) \in L^1([0, \infty), R^+) \cap C[[0, \infty), R^+]$.

If the zero solution of (1) *is uniform-asymptotically stable, then the zero solution of* (2) *is also uniform-asymptotically stable.*

Proof. Set $F(t, u) \equiv a(t)u$, $u \geqslant 0$. Then conditions (1a), (1b), (2a), and (3a) are satisfied. First of all, the zero solution of (2) is uniformly stable by Corollary 1. Next, since the zero solution of (4) is uniformly stable, by Lemma 2, it is $T(m)$-uniform-asymptotically stable. Therefore by Theorem 3, the zero solution of (2) is quasi-uniform-asymptotically stable, which completes the proof of the corollary.

COROLLARY 3.2. *Let the following condition hold for the differential equation* (2):

(2d) *there exist a sufficiently small* $L > 0$ *and a* $\omega > 0$ *such that*

$$\|f(t, x)\| \leqslant L \|x\| \qquad \text{for any } x \in B_\omega.$$

If the zero solution of (1) *is uniform-asymptotically stable, then the zero solution of* (2) *is also uniform-asymptotically stable.*

Proof. Set $F(t, u) \equiv Lu$, $u \geqslant 0$. The solution $y(t) \equiv y(t; t_1, y_1)$ of (4) is of the form $y_1 e^{KL(t-t_1)}$. Since conditions (1a), (1b), (2a), and (3a) of Theorem 3 are satisfied, first of all we show that the zero solution of (4) is $T(m)$-uniform-asymptotically stable. Since L is sufficiently small, we may assume that $KL < m$. Let a $\delta > 0$ be any fixed real number and let any $\varepsilon > 0$ be given. We may assume that $\varepsilon < \delta$. Set $T(\varepsilon) \equiv [1/(KL - m)] \log(\varepsilon/\delta)$. Thus if $|y_1| < \delta$, then

$$y(t) e^{-m(t-t_1)} < \varepsilon, \; t \geqslant t_1 + T(\varepsilon),$$

which implies that the zero solution of (4) is $T(m)$-uniform-asymptotically stable. Therefore by Theorem 3, the zero solution of (2) is quasi-uniform-asymptotically stable. Next, let $x(t) \equiv x(t; t_1, x_1)$ be a solution of (2) and let $K \|x_1\| < y_1$. Then from the proof of Theorem 3, we obtain that

$$\|x(t)\| < y(t) e^{-m(t-t_1)} = y_1 e^{(KL - m)(t-t_1)}, \qquad t \geqslant t_1.$$

Now since $KL - m < 0$, $\|x(t)\| < y_1 e^{(KL-m)(t-t_1)} < y_1$, $t \geqslant t_1$. Thus for any $\varepsilon > 0$, set $\delta(\varepsilon) \equiv \varepsilon/K$. If $\|x_1\| < \delta(\varepsilon)$, then take a $y_1 > 0$ such that $K\|x_1\| < y_1 < \varepsilon$. Thus $\|x(t)\| < \varepsilon$, which implies that the zero solution of (2) is uniform-stable. Therefore the zero solution of (2) is uniform-asymptotically stable. Hence the proof is complete.

Next we prove asymptotic stability properties. Asymptotic stability definition is given as follows.

DEFINITION 4. The zero solution of the differential equation (3) is said to be

(AS) asymptotically stable if the zero solution of (3) is stable and for any $\varepsilon > 0$ and any $t_0 \geqslant 0$, there exist a $\delta(t_0) > 0$ and a $T \equiv T(\varepsilon, t_0, x(t; t_0, x_0)) > 0$ such that if $\|x_0\| < \delta(t_0)$, then $\|x(t)\| < \varepsilon$ for all $t \geqslant t_0 + T$.

PROPOSITION 1. *Suppose that conditions* (2a) *and* (2d) *hold for the differential equation*

$$dx/dt = A(t)x + h(t, x) + f(t, x), \tag{8}$$

where $f(t, x), h(t, x) \in C[[0, \infty) \times B_\delta, R^n]$, (2e) $\|h(t, x)\| \leqslant \lambda(t)$ *for* $(t, x) \in [0, \infty) \times B_\delta$ *and for any* $\varepsilon > 0$, *there exists a* $T(\varepsilon) > 0$ *such that*

$$\int_{T(\varepsilon)}^{t} e^{ms} \lambda(s)\, ds < \varepsilon \qquad \text{for} \quad t \geqslant T(\varepsilon).$$

Then $\|x(t)\| \to 0$, *as* $t \to \infty$.

Furthermore, if the zero solution of (8) *is stable, then the zero solution of* (8) *is asymptotically stable.*

Proof. Since conditions (2a) and (2d) hold for (8) and moreover

$$x(t) = U(t)\, U^{-1}(T)\, x(T) + \int_T^t U(t)\, U^{-1}(s)(h(s, x(s)) + f(s, x(s)))\, ds,$$

we obtain that for $t \geqslant T \geqslant 0$,

$$\|x(t)\|\, e^{mt} \leqslant Ke^{mT}\, \|x(T)\| + \int_T^t K\lambda(s)\, e^{ms}\, ds + \int_T^t KLe^{ms}\, \|x(s)\|\, ds.$$

Since condition (2e) holds, for a sufficiently large $T > 0$ there exists an $M > 0$ such that $\int_T^t K\lambda(s)\, e^{ms}\, ds < M$. Let $Ke^{mT}\, \|x(T)\| + M < y_1$. Then,

$$\|x(t)\|\, e^{mt} - \int_T^t KLe^{ms}\, \|x(s)\|\, ds < y(t) - \int_T^t KLy(s)\, ds.$$

By Lemma 1, we obtain that $\|x(t)\| < y(t)\, e^{-mt} < y_1\, e^{(KL-m)(t-T)}$, which completes the proof of the corollary since $KL - m < 0$.

THEOREM 4. *Let conditions* (1a), (1b), *and* (1c) *hold for the differential equation* (2). *If the zero solution of the differential equation* (4) *is stable and moreover,*

(4a) $U(t) \to 0$ *as* $t \to \infty$,

then the zero solution of (2) *is asymptotically stable.*

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ and $y(t) \equiv y(t; t_1, y_1)$ be solutions of (2) and (4), respectively. Since the zero solution of (4) is stable, for any fixed $B > 0$ and any $t_1 \geqslant 0$, there exists a $\delta(B, t_1) > 0$ such that if $|y(t_1)| < \delta(B, t_1)$, then $|y(t)| < B$ for all $t \geqslant t_1$. Set $\delta(t_1) \equiv \delta(B, t_1)/K$. If $\|x(t_1)\| <$

$\delta(t_1)$, then take a $y_1 > 0$ such that $K\|x_1\| < y_1 < \delta(B, t_1)$. From the proof of Theorem 1, we have that $\|x(t)\| < y(t)$ for all $t \geqslant t_1$. Thus,

$$\delta(B, t_1) + B > |y_1| + |y(t)| \geqslant \int_{t_1}^{t} KF(s, y(s))\, ds$$

$$\geqslant \int_{t_1}^{t} KF(s, \|x(s)\|)\, ds,$$

which implies that for any $\varepsilon > 0$, there exists a $T \equiv T(\varepsilon, t_1, x(t; t_1, x_1)) > 0$ such that $\varepsilon > \int_T^t KF(s, \|x(s)\|)\, ds$ for all $t \geqslant T$. Thus we obtain that from (5) and condition (1a),

$$\|x(t)\| \leqslant \|U(t)\|\ \|U^{-1}(t_1)x_1\| + \|U(t)\| \left\| \int_{t_1}^{T} U^{-1}(s)\, f(s, x(s))\, ds \right\|$$

$$+ \int_T^t KF(s, \|x(s)\|)\, ds,$$

which implies that $\|x(t)\| \to 0$ as $t \to \infty$ by condition (4a). Now by Theorem 1, the zero solution of (2) is stable. Therefore the proof of the theorem is complete.

COROLLARY 4. *Let conditions* (1c), (1d), *and* (1e) *hold for the differential equation* (2). *If condition* (4a) *holds, then the zero solution of* (2) *is asymptotically stable.*

Finally we discuss the quasi-equiasymptotic stability of the zero solution of the perturbed differential equation of a periodic linear differential equation.

COROLLARY 2.2. *Let conditions* (1a), (1b), *and* (2b) *hold. Furthermore the following conditions are satisfied*:

(5a) $A(t) = A(t + \omega)$, $t \geqslant 0$, *where an* ω (>0) *is a period of* $A(t)$,

(5b) *all solutions of the differential equation* (1) *converge to zero as* $t \to \infty$.

If the zero solution of (4) *is* $T(m)$*-asymptotically stable, then the zero solution of the differential equation* (2) *is quasi-equiasymptotically stable.*

Proof. Let $U(t)$ be a fundamental solution matrix of (1). By condition (5a), using Floquet's theorem, we have a non-singular continuous ω-periodic $n \times n$ matrix $V(t)$ and a constant $n \times n$ matrix B such that $U(t) = V(t)\, e^{tB}$, $t \geqslant 0$. From (5b), $\|e^{tB}\| \to 0$ as $t \to \infty$, which implies that there exist suitable positive real numbers $\alpha > 0$ and $\beta > 0$ such that $\|e^{tB}\| \leqslant$

$\alpha e^{-\beta t}$. Since there exists a $\gamma > 0$ such that $\|V(t)\|$, $\|V^{-1}(t)\| \leqslant \gamma$ for all $t \geqslant 0$, we have an $m > 0$ and a $K \geqslant 1$ such that

$$\|U(t)\,U^{-1}(s)\| \leqslant Ke^{-m(t-s)}, \qquad t \geqslant s \geqslant 0.$$

Therefore since all conditions of Theorem 2 are satisfied, the proof of the theorem is complete.

As corollary of this property, we present the following property [1, p. 86].

COROLLARY 2.3. *Suppose that conditions* (2d), (5a), *and* (5b) *hold for the differential equation* (2). *Then the zero solution of the differential equation* (2) *is uniform-asymptotically stable.*

Proof. From the proof of Corollary 2.2, the zero solution of (1) is uniform-asymptotically stable. Therefore the proof is complete by Corollary 3.2.

Remark. From conditions (5a) and (5b), it is induced that the zero solution of (2) is uniformly stable, which is a known property.

4. BOUNDEDNESS THEOREMS

In this section we present boundedness theorems of the differential equation (2).

THEOREM 5. *Let conditions* (1a), (1b), *and* (1c) *hold except that* $F(t, 0) \equiv 0$, *for the differential equation* (2). *If the solutions of the differential equation* (4) *are uniformly bounded* (*bounded, uniform-ultimately bounded*), *then the solutions of* (2) *are also uniformly bounded* (*bounded, uniform-ultimately bounded*).

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ be the solution of (2) and let $y(t) \equiv y(t; t_1, y_1)$ be the solution of (4). Let $K\|x(t_1)\| < y_1$. Then we obtain that in the same way as in the proof of Theorem 1,

$$\|x(t)\| < y(t), \qquad t \geqslant t_1.$$

Since the solutions of (4) are uniformly bounded, for any $\rho > 0$, there exists a $B_0(\rho) > 0$ such that if $|y_1| < \rho$, then $|y(t)| < B_0(\rho)$ for all $t \geqslant t_1$. Thus set $B(\rho) \equiv B_0(K\rho)$. If $\|x(t_1)\| < \rho$, then take a $y_1 > 0$ such that $K\|x_1\| < y_1 < K\rho$. Hence, we obtain that $\|x(t)\| < y(t) < B(\rho)$ for all $t \geqslant t_1$, which completes the proof of the theorem.

In a way similar to that of the proof of Corollary 1, we obtain the following corollary [1, p. 34].

COROLLARY 5. *Let conditions* (1d), (1e), *and* (1c) *hold. Then the solutions of* (2) *are uniformly bounded.*

Next we shall give the following $T(m)$-boundedness definition for a positive real number m.

DEFINITION 5. Let an $m>0$ denote a positive real number. The solutions of the differential equation (3) are said to be $T(m)$-ultimately bounded if there exists a $B>0$, and for any $\rho>0$ and any $t_1\geqslant 0$, there exists a $T(t_1,\rho)>0$ such that

$$\| y(t)\, e^{\,m(t-t_1)}\| < B, \qquad t\geqslant t_1+T(t_1,\rho) \text{ provided } \| y(t_1)\| < \rho.$$

The solutions of (3) are said to be $T(m)$-uniform-ultimately bounded if the above $T(t_1,\rho)$ is independent of time t_1.

LEMMA 8. *If the solutions of* (3) *are uniformly bounded, then they are $T(m)$-uniform-ultimately bounded.*

Proof. Let a $B>0$ be any fixed positive real number. Thus let any $\rho>0$ be given. By the assumption, there exists a $\beta(\rho)>0$ such that if $\| y(t_1)\|<\rho$, then $\| y(t)\|<\beta(\rho)$ for all $t\geqslant t_1$. We assume that $B<\beta(\rho)$. Set $T(\rho)\equiv (1/m)\log(\beta(\rho)/B)$. Hence if $\| y(t_1)\|<\rho$, then $\| y(t)\, e^{-m(t-t_1)}\|<B$, $t\geqslant t_1+T(\rho)$, which completes the proof of the lemma.

LEMMA 9. *If the solutions of* (3) *are uniform-ultimately bounded, then they are $T(m)$-uniform-ultimately bounded.*

LEMMA 10. *If the solutions of* (3) *are equibounded (equiultimately bounded), then they are $T(m)$-ultimately bounded.*

THEOREM 6. *Let conditions* (1a), (1b), (2a), *and* (2b) *hold except that* $F(t,0)\equiv 0$, *for the differential equation* (2). *If the solutions of the differential equation* (4) *are $T(m)$-ultimately bounded, then the solutions of* (2) *are equiultimately bounded.*

Proof. Let $x(t)\equiv x(t;t_1,x_1)$ and $y(t)\equiv y(t;t_1,y_1)$ be the solutions of (2) and (4), respectively. Let any $\rho_0>0$ and any $t_1\geqslant 0$ be given. By the assumption of the theorem, there exist a $B>0$ and a $T_0(t_1,\rho_0)>0$ such that $| y(t)\, e^{-m(t-t_1)}|<B$, $t\geqslant t_1+T_0(t_1,\rho_0)$ provided $| y(t_1)|<\rho_0$. Hence, let $Ke^{m(t_1)} \|x_1\| < y_1$. Then we obtain, in the same way as that of the proof of Theorem 2, that

$$\|x(t)\| < y(t)\, e^{-mt}, \qquad t \geqslant t_1.$$

If $\|x(t_1)\| < \rho$, then take a $y_1 > 0$ such that $Ke^{m(t_1)} \|x_1\| < y_1 < Ke^{m(t_1)}\rho$. Thus set $T(t_1, \rho) \equiv T_0(t_1, Ke^{m(t_1)}\rho)$. Then,

$$\|x(t)\| < y(t)\, e^{-m(t-t_1)} < B, \qquad t \geqslant t_1 + T(t_1, \rho),$$

which completes the proof of the theorem.

THEOREM 7. *Let conditions* (1a), (1b), (2a), *and* (3a) *hold for the differential equation* (2). *If the solutions of the differential equation* (4) *are $T(m)$-uniform-ultimately bounded, then the solutions of* (2) *are uniform-ultimately bounded.*

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ and $y(t) \equiv y(t; t_1, y_1)$ be the solutions of (2) and (4), respectively. Let $K \|x_1\| < y_1$. From the proof of Theorem 3, we have that $\|x(t)\| < y(t)\, e^{-m(t-t_1)}$ for all $t \geqslant t_1$. By the assumption that the solutions of (4) are $T(m)$-uniform-ultimately bounded, there exists a $B > 0$, and for any $\rho_0 > 0$, there exists a $T(\rho_0) > 0$ such that $|y(t)\, e^{-m(t-t_1)}| < B$ for all $t \geqslant t_1 + T(\rho_0)$ provided $|y(t_1)| < \rho_0$. Now let any $\rho > 0$ be given. If $\|x_1\| < \rho$, then take a $y_1 > 0$ such that $K \|x_1\| < y_1 < K\rho$. Thus,

$$\|x(t)\| < y(t)\, e^{-m(t-t_1)} < B, \qquad t \geqslant t_1 + T(K\rho),$$

which completes the proof of the theorem.

Now we need the following elementary proposition [22, p. 45].

PROPOSITION 2. *The zero solution of* (1) *is uniform-asymptotically stable if and only if the solutions of* (1) *are uniform-ultimately bounded.*

As the corollaries of Theorem 7, we prove the following properties.

COROLLARY 7.1. *Let conditions* (1d) *and* (1e) *hold for the differential equation* (2). *If the solutions of* (1) *are uniform-ultimately bounded, then the solutions of* (2) *are also uniform-ultimately bounded.*

Proof. Set $F(t, u) \equiv a(t)u$, $u \geqslant 0$. Then the solutions of (4) are uniformly bounded, which by Lemma 8 and by Theorem 7 completes the proof of the corollary.

COROLLARY 7.2. *Let condition* (2d) *hold for the differential equation* (2). *If the solutions of* (1) *are uniform-ultimately bounded, then the solutions of* (2) *are also uniform-ultimately bounded.*

Proof. Set $F(t, u) \equiv Lu$, $u \geqslant 0$. Since L is sufficiently small, we may assume that $KL < m$. Let $y(t) \equiv y(t; t_1, y_1)$ be the solution of (4) and let a $B > 0$ be any fixed positive real number. For any $\rho > 0$, we may assume that $B < \rho$. Now set $T(\rho) \equiv (1/(KL - m)) \log(B/\rho)$. Then,

$$|y(t)\, e^{-m(t-t_1)}| = |y_1\, e^{(KL-m)(t-t_1)}| < B, \qquad t \geqslant t_1 + T(\rho)$$

provided $|y_1| < \rho$, which implies that the solutions of (4) are $T(m)$-uniform-ultimately bounded. Therefore by Theorem 7, the proof of the corollary is complete.

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